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ANALYSIS OF STRESS INTENSITY FACTORS IN AN OSCILLATING PLATE WITH A CRACK BY THE METHOD OF FINITE ELEMENTS

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The factor in the singular part of solution of the problem of longitudinal oscillations of a plate with a straight crack is determined by the method of finite elements with the introduction of a special finite element that takes into account the singularity at the crack edge.

The method of finite elements was used in dynamic problems of linear fracture mechanics due to stationary crack in connection with the investigation of shock wave diffraction by a longitudinal shear crack [1]; it proved to result in a considerable error. The method of difference schemes [2] applied to similar nonstationary problems yielded more accurate results. It is shown below that in vibration problems the method of finite elements is effective.

1. Let us assume that a body with a crack is subjected in addition to harmonic force components to static tension in such a way that the crack edges do not overlap. Since displacement are specified along a part of the boundary, the matrix of the system rigidity is not degenerate. This is of no importance in statics, since displacements in a rigid body as a whole do not affect the intensity coefficients, but is a necessary condition when the mode of free oscilations is to be determined. Although in the latter case analytic solutions are not available, the accuracy of computations can be tested by other means, as will be shown below.

The equations of motion for an elastic body free of damping subjected to harmonic loading are of the form

$$M\mathbf{x}^{\cdot\cdot} + K\mathbf{x} = \mathbf{f}\cos\omega t \tag{1.1}$$

where M and K are matrices of mass and rigidity, respectively, $\mathbf{x}$ is the displacement vector of the system, and $\mathbf{f}$ the load vector. When $\omega = 0$, we have the equations of equilibrium

$$K\mathbf{x} = \mathbf{f} \tag{1.2}$$

We denote by ω_i^2 the eigenvalues in their ascending order and by $\mathbf{x}^{(i)}$ the eigenvectors of the generalized problem in eigenvalues

$$K\mathbf{x} = \omega^2 M\mathbf{x}, \quad \mathbf{x}^{(i)T} M\mathbf{x}^{(j)} = \delta_{ij} \tag{1.3}$$

When $\omega \neq \omega_i$ and $\mathbf{x}(0) = 0$ the general solution of Eq. (1.1) is

$$\mathbf{x}(t) = \sum \mathbf{x}^{(i)}\left[\frac{(\mathbf{x}^{(i)T}\mathbf{f})}{\omega_i^2 - \omega^2}\cos\omega t + \beta_i\cos\omega_i t\right] \tag{1.4}$$

where β_i are undetermined coefficients. To test the validity of (1.4) it is sufficient to substitute it into (1.1), multiply successively by $\mathbf{x}^{(i)T} M$ where the subscript T denotes transposition, and use (1.3) [3].

We similarly obtain the static solution

$$\mathbf{x}_* = \sum \frac{(\mathbf{x}^{(i)T}\mathbf{f})}{\omega_i^2} \mathbf{x}^{(i)} \tag{1.5}$$

The displacement vector of an elastic body uniquely defines the intensity coefficients by a linear functional.

We denote by K_* the static intensity coefficient that corresponds to $\mathbf{x}_*$, and by $K^{(i)}$ the intensity coefficients which correspond to $\mathbf{x}^{(i)}$. The dimension of $K^{(i)}$ is determined with allowance for (1.3).

We introduce the dimensionless coeffcients

$$\alpha_i = \frac{K^{(i)}(\mathbf{x}^{(i)T}\mathbf{f})}{K_*\omega_i^2}, \quad K_* = \sum K^{(i)} \frac{(\mathbf{x}^{(i)T}\mathbf{f})}{\omega_i^2} \tag{1.6}$$

From (1.4) and (1.5) we have

$$\varkappa(t) = \frac{K(t)}{K_*} = \sum \left(\alpha_i \frac{\omega_i^2}{\omega_i^2 - \omega^2} \cos \omega t + \beta_i' \cos \omega_i t\right) \tag{1.7}$$

where $(K(t)$ is the dynamic intensity coefficient under the condition

$$\sum \alpha_i = 1 \tag{1.8}$$

Condition (1.8) may be taken as the criterion of accuracy of the dynamic intensity coefficient computation, provided that the static intensity factor K_* appearing in the expression for α_i in (1.6) is not computed by formula (1.5) and the second of formulas (1.6), but directly using the static system of equilibrium equations (1.2). Moreover condition (1.8) shows in the frequency region $\omega < \omega_1$ the number of oscillation modes that are to be taken into account in (1.7).

Thus the dynamic intensity factors are determined with an error defined by the remainder $|\Sigma\alpha_i - 1|$. In other words, the error of determination of $K(t)$ can be estimated by comparing the dynamic intensity coefficient at zero frequency with that of static intensity calculated by the equilibrium equations. This follows from the equivalence of the second of equalities (1.6) and equality (1.8). Numerical computations had shown that in the low frequency region it is sufficient to take into account the first eight oscillation modes, which results in an error in the calculated dynamic intensity coefficient not exceeding 6%. It appears that in the case of loading leading to normal rupture at the crack there always are two fairly large positive coefficients α_i, $i < 3$, while negative α_i are small in modulus and occur only for $i \geqslant 3$, consequently, the quantity

$$\left|\sum \alpha_i \frac{\omega_i^2}{\omega_i^2 - \omega^2}\right| > 1$$

and increases as ω increases from zero to ω_1.

If induced free oscillations are taken into account, the intensity factor amplitude may increase even further. For instance, let $\mathbf{x}(0) = 0$, then

$$\beta_i' = -\alpha_i \frac{\omega_i^2}{\omega_i^2 - \omega^2}$$

Since ω_i are, as a rule, high, functions $\cos \omega_i t$ may in a short time interval simultaneously reach extreme values with the same signs as those of β_i'. The maximum of $\varkappa(t)$ is then additionally defined by the sum

$$\sum |\alpha_i| \frac{\omega_i^2}{\omega_i^2 - \omega^2} > \left| \sum \alpha_i \frac{\omega_i^2}{\omega_i^2 - \omega^2} \right| > 1$$

For the determination of the singular stress fields at the crack it is necessary to considerably reduce the diameter of finite elements close to the crack. However this leads to an increase of the order of related equations, and makes the computation ineffective. This can be avoided by placing at the crack tip a singular finite element whose strain field is approximated on the basis of analytic solutions for a region with a crack. Here, the construction of the singular element is based on the expansion of the strain field in the crack tip neighborhood [4]. The universality of such expansions was shown in [5]. Derivation of the rigidity matrix for the singular element and examples of its application in plane static problems is described in [6]. The accuracy obtainable when using such finite elements is fairly high, in spite of the incompatibility of the strains at the interface with conventional finite elements. In the case of short cracks the discrepancy between the computed stress intensity factor and the analytic static solution for such cracks does not exceed 2%. A similar method was used in [7, 8] in problems of bending.

In the case of matrices of higher order the generalized problem in eigenvalues (1.3) cannot be reduced to the conventional problem with a single matrix (inversion of matrices leads to considerable errors), and it is not possible to reduce the order of matrices because of the necessity of exact determination of eigenvectors. Methods that use specific aspects of a problem (the band property of matrices, possibility of determining only part of the spectrum) have recently appeared (methods of simultaneous iterations) [9]. First, we reduce (1.3) to the equivalent problems with a single matrix.

Let

$$K = LL^T, \quad \lambda = 1 / \omega^2, \quad \mathbf{x} = L^T \mathbf{p}$$

then instead of (1.3) we have

$$A\mathbf{x} = \lambda \mathbf{x}, \quad A = L^{-1} M L^{-T} \tag{2.1}$$

The multiplication $L^{-1}ML^{-T}\mathbf{x}$ is carried out in three stages

$$\begin{aligned} &L^T\mathbf{u} = \mathbf{x}, \quad \mathbf{u} = L^{-T}\mathbf{x} \\ &\mathbf{y} = M\mathbf{u}, \quad \mathbf{y} = ML^{-T}\mathbf{x} \\ &L\mathbf{v} = \mathbf{y}, \quad \mathbf{v} = L^{-1}ML^{-T}\mathbf{x} \end{aligned}$$

Let the eigenvalues in (2.1) be $|\lambda_1| \geqslant |\lambda_2| \geqslant \ldots \geqslant |\lambda_n|$. If it is required to determine the first m eigenvalues, matrix $U = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_q]$ $(m < q \ll n)$ is used for computations.

The iteration process consists of six stages: $U^T U = I$ (orthogonalization), $V = AU$, $B = U^T V$, $B\mathbf{t}^* = \lambda \mathbf{t}^*$ (exact or approximate determination of eigenvectors

$t_1^*, \ldots, t_q^*$ of matrix B), and $W = VT^*$ ($T^* = [t_1^*, t_2^*, \ldots, t_q^*]$), $U = W$.

3. As an illustration we shall consider a rectangular plate with an edge crack in a plane state of stress. One of the plate edges of length b is restrained, while the opposite edge is subjected alternatively to tension and compression of the same intensity in conformity with a harmonic law. The plate has a crack of length l parallel to these sides located at the middle line of the plate whose other sides are of length a.

Table 1

	$l/b=0.083$	0.167	0.250	0.333	0.417
1	0.32	0.30	0.26	0.25	0.16
2	2.05	1.69	1.27	1.12	0.25
3	2.54	2.40	2.39	2.36	2.36
4	9.02	8.40	7.71	7.19	5.76
5	9.29	9.26	8.95	8.75	8.57
6	11.54	11.41	11.26	11.23	10.74
7	14.74	13.70	12.90	12.44	11.44
8	18.88	18.05	15.66	14.08	13.54

Squares of dimensionless frequencies of free oscillations $\omega_{nd}^2 = \omega^2 \rho ab / E$, where ρ is the plate density, and E is the Young's modulus are shown in Table 1, for a number of relative lengths of cracks, and a Poisson ratio equal 0.3. Computations were carried out assuming a uniform distribution of elements, and the number of nodes varied from 58 to 66, depending on the crack length. Altogether 16 vibration modes were determined.

Curves showing the dependence of maximum of $\varkappa(t) = K(t) / K_*$ defined in (1.7) on ω_{nd}^2 appear in Fig. 1 for relative crack lengths 0.167, 0.25, and 0.417 (curves *1*–*3*, respectively).

The dynamic intensity coefficient for a given amplitude of the applied stress σ can be obtained from these curves and the static intensity coefficient K_* which is of the form $K_* = \sigma \sqrt{\pi l}\, F(l / b,\, a / b)$. In this example computations were based on a ratio of plate sides $a / b = 1.17$. Function $F(l / b,\, a / b)$ for the investigated crack lengths was equal 1.40, 1.65, and 2.53 respectively.

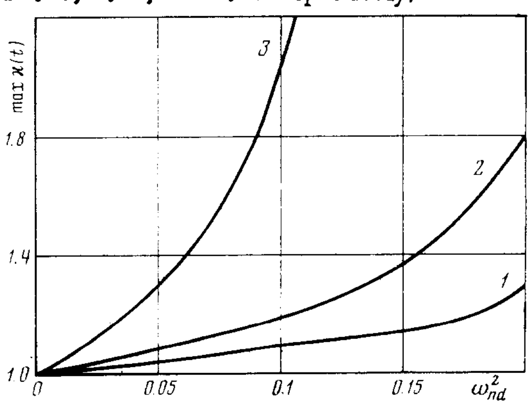

Fig. 1

The shape of curves *1—3* implies that the danger of brittle fracture increases with increasing frequency of load [reversal]. A similar result was obtained in [10] in connection with the solution of the problem of oscillations of a plane with a system of cracks and of an infinite strip with a crack. Hence the design of structural elements subjected to high-frequency loading must be aimed at increasing the natural frequency of their oscillations. Certain conclusions about the magnitude of frequencies can be arrived at on the basis of corollaries of the Courant—Fischer theorem [3]. Thus, when the oscillation frequencies of a plate containing a crack are $\omega_1 \leqslant \omega_2 \leqslant \ldots$ and the frequency of a plate with a shorter crack, which in finite element discretization may be considered equal to the first one, except for r relations imposed on it, have the frequencies $\omega_1' \leqslant \omega_2' \leqslant \ldots$, then that theorem implies that $\omega_i \leqslant \omega_i' \leqslant \omega_{i+r}$.

Thus for the evaluation of free oscillation limits it is sufficient to determine the frequencies of a plate containing a crack or a system of crack of maximum length. On the basis of the Courant—Fischer theorem it is possible to state that an increase of oscillation frequencies can be obtained by reducing the length of cracks, securing part of the boundary, reducing [the extent of] plastic zones, reducing the structure mass and increasing its rigidity (e.g., by adding stiffening ribs), etc.

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